

# On the interpretation of the equatorially antisymmetric Jovian gravitational field

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## ABSTRACT

Since the odd zonal gravitational coefficients of Jupiter are nearly unaffected by the planet’s rotational distortion, an effective way of estimating the internal structure of the equatorially antisymmetric Jovian winds is to measure the odd coefficients induced by their equatorially antisymmetric component and then apply a mathematical theory to “invert” them. The thermal-gravitational wind equation (TGWE) provides this theoretical basis for interpretation. Here we show that the kernel term of the TGWE requires that its solutions satisfy a solvability condition. The thermal wind equation is a diagnostic relation that generates a “solution” for any zonal wind profile, but that “solution” does not necessarily satisfy the solvability condition required for the TGWE. We develop a new approach to solving the TGWE that respects the solvability condition. We then calculate the odd zonal gravitational coefficients of Jupiter using a profile of zonal winds that satisfies the solvability condition and is equatorially antisymmetric and consistent with the observed cloud-level winds of Jupiter. We also explain the subtle but profound difference between the TWE and the TGWE via an analogous inhomogeneous ordinary differential equation. The developed method can be readily extended for inversion of the data soon to be acquired by the Juno spacecraft.

## Key words:

Rotating planet and star, shape, internal structure, gravitational field.

## 1 INTRODUCTION

The external zonal gravitational potential  $V_g$  of Jupiter can be expanded in terms of the Legendre functions  $P_n$ ,

$$V_g = -\frac{GM_J}{r} \left\{ 1 - \sum_{n=1}^{\infty} [J_n + \Delta J_n] \left( \frac{R_J}{r} \right)^n P_n(\theta) \right\}, \quad (1)$$

where  $M_J$  is Jupiter’s mass,  $(r, \theta)$  are spherical polar coordinates with  $\theta = 0$  being at the axis of rotation,  $R_J$  is the equatorial radius of Jupiter,  $r > R_J$ ,  $G$  is the universal gravitational constant ( $G = 6.67384 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$ ),  $n$  takes integer values, and  $(J_1 + \Delta J_1), (J_2 + \Delta J_2), (J_3 + \Delta J_3), \dots$  denote the zonal gravitational coefficients to be measured by the Juno spacecraft (Bolton 2005).

It is important to understand that there are fundamental differences between the even and odd zonal gravitational

coefficients in (1). For the even coefficients  $(J_{2k} + \Delta J_{2k})$  with  $k \geq 1$ , the rotational distortion gives rise to  $J_{2k}$  while their corrections  $\Delta J_{2k}$  are produced by the fast equatorially symmetric zonal winds (Hubbard 1999; Kaspi et al. 2010; Kong et al. 2013; Liu et al. 2013). Identification of a small correction  $\Delta J_{2k}$  from the measured gravitational coefficient  $(J_{2k} + \Delta J_{2k})$  is a highly difficult problem. For the odd gravitational coefficients  $(J_{2k+1} + \Delta J_{2k+1})$  with  $k \geq 1$  in (1), however, there is no such difficulty. This is because the rotational distortion, owing to its equatorial symmetry, does not contribute to the odd coefficients (*i.e.*,  $J_{2k+1} = 0$  with  $k \geq 1$ ) and, consequently, the size of the odd coefficients directly reflects the structure/amplitude of the equatorially antisymmetric zonal winds in the interior of Jupiter. This is also because the size of the odd coefficients, as suggested by our calculation in this paper, is much larger than the noise level and, hence, should be accurately measured by the Juno spacecraft (Bolton 2005). A theoretical relationship between

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the equatorially antisymmetric winds and the wind-induced odd coefficients  $\Delta J_{2k+1}$ , together with the coefficients to be accurately determined by the high-precision measurements of the Juno spacecraft, will enable estimation of the deep structure of the Jovian cloud-level winds.

Two different attempts have been made to estimate the odd gravitational coefficients  $\Delta J_{2k+1}$ ,  $k = 1, 2, 3$ , induced by the equatorially antisymmetric winds of Jupiter. Kaspi (2013) computed the odd coefficients in spherical geometry on the basis of the thermal wind equation using the equatorially antisymmetric zonal winds  $U_A$  in the form

$$U_A(r, \theta, H) = u_0(r \sin \theta) e^{-(R_J - r)/H}, \quad \text{for } 0 \leq \theta < \pi/2 \quad (2)$$

where  $r \sin \theta$  denotes the distance from the rotation axis, and  $H$  is a depth parameter characterizing the attenuation of the zonal winds in the interior.  $u_0(r \sin \theta)$  represents the equatorially antisymmetric component of the observed cloud-level zonal winds (Porco et al. 2003) in the northern hemisphere extending into the equator on cylinders parallel to the rotation axis. There are two deficiencies in the thermal-wind-equation model. First,  $U_A$  given by (2) is discontinuous across the equatorial plane which, as pointed out by Kong et al. (2016), makes a non-physical contribution to the odd gravitational coefficients that is dominant when the winds are deep. Second, and perhaps more important, the thermal wind equation with the wind profile (2) does not satisfy, for any value of  $H$ , a solvability condition shown below to be a requirement for a mathematically valid solution. Kong et al. (2015) computed the odd gravitational coefficients by solving the full governing equations of the problem in the northern hemisphere of Jupiter with the equatorially anti-symmetric condition required at the equatorial plane explicitly imposed. There are also shortcomings in the hemispheric model. The method is valid only at the limit  $H \rightarrow \infty$  in (2) and the model has to introduce the discontinuity of the winds across the equatorial plane which is non-physical. In short, both the thermal-wind-equation and hemispheric models are inadequate for interpreting the Jupiter's equatorially antisymmetric gravitational field.

Although progress has been made in modeling the zonal winds (for example, Jones and Kuzanyan 2012; Gastine and Wicht 2012) and the evolution and internal structure of Jupiter-like planets (for example, Helled et al. 2013), achieving the realistic physical parameters is difficult and extrapolating the solutions from a numerically accessible model over many orders of magnitude may be unreliable. High-precision gravitational measurements being carried out by the Juno spacecraft provide an alternative way of probing the interior fluid motion and structure of Jupiter. The primary objectives of this paper are twofold: to explain why the thermal wind equation (the TWE) with (2) may lead to a spurious interpretation of the the equatorially antisymmetric gravitational data expected from the Juno mission and to provide, based on the thermal-gravitational wind equation (the TGWE), an estimate of the Jovian odd zonal gravitational coefficients which can be used for the interpretation of the equatorially antisymmetric gravitational field of Jupiter.

## 2 TGWE VS. TWE APPROACH

Our model is based on the following assumptions: (i) Jupiter with mass  $M_J$  and radius  $R_J$  is isolated and rotating about the symmetry  $z$ -axis with an angular velocity  $\Omega \hat{z}$ ; (ii) the effect of the rotational distortion on estimating the odd gravitational coefficients can be neglected (Kaspi 2013); (iii) Jupiter is axially symmetric and consists of a compressible barotropic fluid (a polytrope of index unity) whose density  $\rho$  is a function only of the pressure  $p$ , i.e.,  $p = K \rho^2$  with  $K$  being a constant (Hubbard 1999), and (iv) the cloud-level zonal winds of Jupiter have an equatorially antisymmetric component  $U_{asym}$  that may penetrate into its deep interior. Spherical geometry is adequate for estimating the odd zonal gravitational coefficients of Jupiter (Kong et al. 2016). Moreover, as discussed by Zhang et al. (2015), the spherical geometry assumption implies that  $\Omega$  is small such that the term  $(\rho' \Omega^2 / 2) \nabla |\hat{z} \times \mathbf{r}|^2$  should be neglected.

The TGWE (Zhang et al. 2015) – which describes a relationship between the equatorially antisymmetric winds  $U_{asym}$  and the wind-induced density anomaly  $\rho'$  in spherical geometry – can be written in the form

$$\begin{aligned} & \frac{\rho'(r, \theta) g_0(r)}{r} - \frac{2\pi G (d\rho_0/dr)}{r} \\ & \times \int_0^\pi \int_0^{R_J} \frac{\tilde{r}^2 \rho'(\tilde{r}, \tilde{\theta})}{|\mathbf{r} - \tilde{\mathbf{r}}|} \sin \tilde{\theta} d\tilde{r} d\tilde{\theta} \\ & = 2\Omega \int_{\pi/2}^\theta \left[ \cos \tilde{\theta} \frac{\partial}{\partial r} - \frac{\sin \tilde{\theta}}{r} \frac{\partial}{\partial \tilde{\theta}} \right] \left[ \rho_0(r) U_{asym}(r, \tilde{\theta}) \right] d\tilde{\theta}, \end{aligned} \quad (3)$$

where the density  $\rho_0(r)$  and the gravity  $g_0(r)$  denote the hydrostatic state of the gaseous planet,  $\mathbf{r} = \mathbf{r}(r, \theta)$ ,  $\tilde{\mathbf{r}} = \tilde{\mathbf{r}}(\tilde{r}, \tilde{\theta})$ , and the second term on its left side, a two-dimensional kernel integral with the Green's function in its integrand, represents the gravitational perturbation caused by the density anomaly  $\rho'$ . In general, we have

$$\frac{|g_0 \rho'|}{\left| G (d\rho_0/dr) \int_0^\pi \int_0^{R_J} \left( \tilde{r}^2 \rho'(\tilde{r}, \tilde{\theta}) / |\mathbf{r} - \tilde{\mathbf{r}}| \right) \sin \tilde{\theta} d\tilde{r} d\tilde{\theta} \right|} = O(1).$$

It follows that the second term on the left side of (3) should be generally retained; see the relevant discussion and computation in Zhang et al. (2015). The TWE adopted by Kaspi (2013) is obtained by neglecting the two-dimensional kernel integral in (3), which gives rise to

$$\begin{aligned} \rho'(r, \theta) &= \frac{2r\Omega}{g_0(r)} \\ & \times \int_{\pi/2}^\theta \left[ \cos \tilde{\theta} \frac{\partial}{\partial r} - \frac{\sin \tilde{\theta}}{r} \frac{\partial}{\partial \tilde{\theta}} \right] \left[ \rho_0(r) U_{asym}(r, \tilde{\theta}) \right] d\tilde{\theta}. \end{aligned} \quad (4)$$

where  $\rho_0(r)$  and  $g_0(r)$  are the same as in (3).

The TWE (4) is not simply an approximation to the TGWE (3). There are mathematically profound differences between the two equations. In the case of the TWE, there always exists a solution for the density anomaly  $\rho'$ , the left side of (4), for any given function  $U_{asym}(r, \theta)$ . In the case of the TGWE, we have to solve a two-dimensional inhomogeneous integral equation to obtain the density anomaly  $\rho'$  and, hence, the existence of a solution  $\rho'$  to (3) depends on the form of  $U_{asym}(r, \theta)$ . Obviously, if solutions  $\rho'$  to (3) for a given  $U_{asym}$  do not exist, the TWE (4) using the given  $U_{asym}$  cannot provide an approximation to the TGWE (3) and, consequently, the solutions from the TWE (4) in this

special case are both mathematically and physically meaningless when they are used for computing the wind-induced odd coefficients  $\Delta J_{2k+1}$ .

### 3 WHY TGWE AND TWE LEAD TO FUNDAMENTALLY DIFFERENT SOLUTIONS

We first explain why the TGWE (3), a two-dimensional integral equation, cannot be approximated by the TWE (4), together with (2), for computing  $\Delta J_{2k+1}$ . Since  $g_0(r) = 2K \, d\rho_0/dr$ , the TGWE (3) can be written in the form

$$\mathcal{F}(U_{asym}) = \rho'(r, \theta) - \frac{\pi G}{K} \int_0^\pi \int_0^{R_J} \frac{\tilde{r}^2 \rho'(\tilde{r}, \tilde{\theta})}{|\mathbf{r} - \tilde{\mathbf{r}}|} \sin \tilde{\theta} \, d\tilde{r} \, d\tilde{\theta}, \quad (5)$$

where  $\mathcal{F}(U_{asym})$ , the inhomogeneous term of the integral equation, is a function of the equatorially antisymmetric winds  $U_{asym}(\theta, r)$ . It is well-known (Corduneanu 1991) that solutions of the inhomogeneous integral equation (5) exist if and only if the inhomogeneous term  $\mathcal{F}(U_{asym})$  is orthogonal to every eigenfunction  $\Phi(r, \theta)$  that satisfies

$$0 = \Phi(r, \theta) - \frac{\pi G \lambda}{K} \int_0^\pi \int_0^{R_J} \frac{\tilde{r}^2 \Phi(\tilde{r}, \tilde{\theta})}{|\mathbf{r} - \tilde{\mathbf{r}}|} \sin \tilde{\theta} \, d\tilde{r} \, d\tilde{\theta}, \quad (6)$$

where equation (5) suggests an eigenvalue of the integral equation  $\lambda = 1$ . We can show that the eigenfunction  $\Phi(r, \theta)$  for (6) at  $\lambda = 1$  is given by

$$\Phi(r, \theta) = \frac{1}{r^2} \left( \sin \frac{r\sqrt{2\pi G}}{\sqrt{K}} - \frac{r\sqrt{2\pi G}}{\sqrt{K}} \cos \frac{r\sqrt{2\pi G}}{\sqrt{K}} \right) \cos \theta. \quad (7)$$

It follows that if

$$\int_0^\pi \int_0^{R_J} \left[ \frac{1}{r^2} \left( \sin \frac{r\sqrt{2\pi G}}{\sqrt{K}} - \frac{r\sqrt{2\pi G}}{\sqrt{K}} \cos \frac{r\sqrt{2\pi G}}{\sqrt{K}} \right) \cos \theta \right] \times \mathcal{F}(U_{asym}) \, r^2 \sin \theta \, dr \, d\theta \neq 0, \quad (8)$$

there exist no solutions to the TGWE (3) and, consequently, numerical solutions of the TGWE (3) would always be divergent. Obviously, this solvability condition only applies to the equatorially antisymmetric winds.

Using the wind profile (2) adopted by (Kaspi 2013), it can be readily demonstrated that

$$\int_0^\pi \int_0^{R_J} \left[ \frac{1}{r^2} \left( \sin \frac{r\sqrt{2\pi G}}{\sqrt{K}} - \frac{r\sqrt{2\pi G}}{\sqrt{K}} \cos \frac{r\sqrt{2\pi G}}{\sqrt{K}} \right) \cos \theta \right] \times \mathcal{F}(U_A) \, r^2 \sin \theta \, dr \, d\theta \neq 0, \quad (9)$$

for any non-zero values of the depth parameter  $H$ . Note that the limit  $H \rightarrow 0$  is physically irrelevant because  $\Delta J_{2k+1} \rightarrow 0$  in the limit  $H \rightarrow 0$ . We have therefore encountered an intriguing and unusual problem. Mathematically, the wind profile  $U_A$  given by (2) always yields a solution  $\rho'$  to the TWE (4) (Kaspi 2013) but there exist no solutions to the TGWE (3) for the same  $U_A$ . Physically, the density anomaly  $\rho'$  obtained from the TWE (4) using  $U_A$  given by (2) is not meaningful because the TWE (4) with  $U_A$  does not approximate the TGWE (3). Numerically, solutions to the TGWE (3) for  $U_A$  given by (2) always diverge because the existence condition for the inhomogeneous integral equation (3) is not satisfied. In Section 5, we shall attempt to explain the subtlety of the problem via a simple inhomogeneous ordinary differential equation.

**Table 1.** The distance  $\Delta z$  between the center of mass and the center of figure  $\Delta z$  and the lowermost odd zonal gravitational coefficients  $J_n, n \geq 3$  in the expansion (1) induced by the equatorially antisymmetric zonal winds  $U_B$  given by (13). The second column represents the odd coefficients obtained from the TGWE (3) while the third column is obtained from the TWE (4). The TWE solution in this table is for a wind profile that satisfies the solvability condition. The TWE approximation is valid only for such a wind profile.

	TGWE	TWE
$\Delta z$ (km) (or $J_1$ )	13.294	0.099
$J_3 \times 10^6$	-1.825	-1.355
$J_5 \times 10^6$	0.306	0.259
$J_7 \times 10^6$	0.490	0.458

### 4 ESTIMATING THE ODD ZONAL COEFFICIENTS $\Delta J_{2K+1}$ OF JUPITER

In order to compute the odd gravitational coefficients  $J_1, J_2, J_5 \dots$  ( $J_1$  represents the displacement of the center of mass from the center of figure) induced by the Jovian equatorially antisymmetric winds, we need to choose a profile of the winds  $U_B(r, \theta)$  that satisfies both the solvability condition required for (3)

$$\int_0^\pi \int_0^{R_J} \left[ \frac{1}{r^2} \left( \sin \frac{r\sqrt{2\pi G}}{\sqrt{K}} - \frac{r\sqrt{2\pi G}}{\sqrt{K}} \cos \frac{r\sqrt{2\pi G}}{\sqrt{K}} \right) \cos \theta \right] \times \mathcal{F}(U_B) \, r^2 \sin \theta \, dr \, d\theta = 0, \quad (10)$$

and the equatorial symmetry condition

$$U_B(r, \theta) = -U_B(r, \pi - \theta) \text{ and } U_B(r, \pi/2) = 0. \quad (11)$$

Since we do not know the internal structure of the equatorially antisymmetric winds in Jupiter, our choice is largely guided by the cloud-level latitudinal profile and the theoretical understanding of rotating flow in spherical geometry. According to the theory of spherical inertial modes (Zhang et al. 2010) – which mathematically form a complete set of functions (Ivers et al. 2015) – an equatorially antisymmetric zonal wind can always be expanded in the form

$$U_B = (r \cos \theta) U_1(r \sin \theta) + (r \cos \theta)^3 U_3(r \sin \theta) + \dots, \quad (12)$$

where  $U_n(r \sin \theta)$  is only a function of  $r \sin \theta$ . We take the leading-order term in the above expansion

$$U_B(r, \theta) = \frac{r \cos \theta}{\sqrt{R_J^2 - (r \sin \theta)^2}} u_0(r \sin \theta), \quad 0 \leq \theta < \pi/2, \quad (13)$$

which is chosen such that  $U_B(r = R_J, \theta)$  represents the cloud-level winds of Jupiter (Porco et al. 2003). The equatorially antisymmetric zonal winds  $U_B$  given by (13) satisfy the solvability condition (10) required for the TGWE (3), obey the equatorial symmetry condition (11) and are consistent with the Jovian cloud-level winds (Porco et al. 2003).

With the equatorially antisymmetric zonal winds  $U_B$  given by (13), we can solve the TGWE (3) for the density anomaly  $\rho'(r, \theta)$  using the method proposed by Zhang et al. (2015). Since solutions of (3) give the wind-induced density anomaly  $\rho'(r, \theta)$  obeying the equatorial parity

$$\rho'(r, \theta) = -\rho'(r, \pi - \theta),$$

the center of mass for Jupiter would slightly shift from the center of figure along the axis of rotation. The distance  $\Delta z$  between the center of mass and the center of figure (which is proportional to the size of  $J_1$ ) can be calculated by the following integration

$$\Delta z = \frac{4\pi}{M_J} \int_0^{\pi/2} \int_0^{R_J} \rho' r^3 \sin \theta \cos \theta \, dr \, d\theta.$$

In other words the center of mass in the presence of the equatorially antisymmetric winds  $U_B$  given by (13) is located at  $(\theta = 0, r = \Delta z)$  on the rotation axis. We also compute the odd zonal gravitational coefficients  $J_n$  with  $n \geq 3$  by performing the two-dimensional integration

$$J_n = -\frac{4\pi}{M_J R_J^n} \int_0^{\pi/2} \int_0^{R_J} \rho'(r, \theta) P_n(\cos \theta) \sin \theta r^{n+2} \, dr \, d\theta,$$

for  $n = 3, 5, 7, \dots$

The results of our computation, the distance  $\Delta z$  and the lowermost odd coefficients  $J_{2k+1}$  based on the TGWE (3) and the TWE (4) both using the wind profile (13) are presented in Table 1. The TWE can be used in this case since the wind profile satisfies the solvability condition. In the case of the TGWE (3), we obtain  $\Delta z = 13.294$  km while the TWE gives  $\Delta z = 0.099$  km. The order-of-magnitude difference between solutions of the TGWE (3) and the TWE (4) reflects the fact that the second term on the left side of (3), a two-dimensional kernel integral, makes a dominant contribution to the large-scale density anomaly  $\rho'$ . *For the small-scale variation of  $\rho'$  associated with the higher-order coefficients  $J_{2k+1}$ , however, it is expected that the integral term would make a less significant contribution because of the effective cancelation of the positive ( $\rho' > 0$ ) and the negative ( $\rho' < 0$ ) density anomaly after averaging over the sphere.* For example, in the case of the TGWE (3), we obtain  $J_5 = 0.306 \times 10^{-6}$  while the TWE (4) gives  $J_5 = 0.259 \times 10^{-6}$ . Even though the TWE (4) in this case provides a reasonable approximation, a consequence of the cancelation effect with the small-scale variation which does not mean that the integration term is unimportant, we still have to use the TGWE (3) to check whether the profile of an equatorially antisymmetric zonal wind satisfies the solvability condition (10). Any results based on the TWE (4) using the wind profile (2) that violate the solvability condition are not compatible with TGWE results and can thus lead to spurious interpretation of gravity data.

The results of our calculation with  $U_B$  given by (13) show that the TGWE (3) is profoundly different from the TWE (4) in two important ways. First, the profile of an equatorially antisymmetric zonal wind must satisfy the solvability condition required for the TGWE (3) but there is no such requirement for the TWE (4). Second, the TWE (4), using  $U_B$  given by (13) satisfying the required solvability condition for the TGWE (3), does not provide an approximation to the TGWE (3) for the lowest harmonics that corresponds to the large-scale variation of the density anomaly  $\rho'$ : there is an order-of-magnitude difference between solutions  $\Delta z$  of the TGWE (3) and the TWE (4).

## 5 EXPLANATION OF SUBTLETY OF THE PROBLEM

Since two-dimensional inhomogeneous integral equations are not only complicated but also infrequently encountered in planetary and astrophysical problems, it is perhaps easier to explain the subtle but profound difference between the TWE (4) and the TGWE (3) via a simple inhomogeneous ordinary differential equation of the form

$$\pi^2 \rho(x) + \frac{d^2 \rho(x)}{dx^2} = f(u(x)) \quad (14)$$

subject to the boundary condition

$$\rho(0) = \rho(1) = 0,$$

where the inhomogeneous term  $f(u(x))$  is non-zero and satisfies  $f(0) = 0$  and  $f(1) = 0$ . In (14),  $\pi^2 \rho(x)$  resembles the first term on the left side of (3),  $f(u(x))$  is similar to the right side of (3), and  $d^2 \rho(x)/dx^2$  is, apart from a differential operator vs. an integral operator, analogous to the second term on the left side of (3). We consider three different cases in order to illustrate the subtlety of the problem.

In the first case, which is analogous to the case of the TWE (4), we make an assumption that even though  $d^2 \rho(x)/dx^2$  is generally of the same order as  $\pi^2 \rho(x)$ , the derivative term in (14) can be neglected and, consequently, the solution of (14) always exists for any given function  $f(u(x))$ , which is

$$\rho(x) = \frac{f(u(x))}{\pi^2}.$$

It follows that any profile  $f(u(x))$  can yield a solution  $\rho(x)$  for the equation (14) that neglects the term  $d^2 \rho(x)/dx^2$ . However, it is unknown whether  $\rho(x) = f(u(x))/\pi^2$  represents an approximation to, or is even relevant to, the true solution of (14).

In the second case, which is analogous to the case of the TGWE (3) together with  $U_A$  given by (2), we include the term  $d^2 \rho(x)/dx^2$  in (14) but assume that  $f(u_A(x))$  does not satisfy the solvability condition, *i.e.*,

$$\int_0^1 \sin(\pi x) f(u_A(x)) dx \neq 0. \quad (15)$$

It follows that solutions to the inhomogeneous equation (14) do not exist and, consequently, numerical solutions of (14) with this particular  $f(u_A(x))$  always diverge if one attempts to solve it numerically.

In the third case, which is analogous to the case of the TGWE (3) together with  $U_B$  given by (13), we include the term  $d^2 \rho(x)/dx^2$  in (14) and assume that  $f(u_B(x))$  satisfies the solvability condition, *i.e.*,

$$\int_0^1 \sin(\pi x) f(u_B(x)) dx = 0. \quad (16)$$

It follows that solutions to the inhomogeneous equation (14) exist and are mathematically meaningful. If one attempts to solve it numerically in this case, the corresponding solution would be convergent.

A central point is that, when  $f(u_A(x))$  does not satisfy the solvability condition, the solution  $\rho(x) = f(u_A(x))/\pi^2$  obtained in the first case not only does not approximate the true solution of (14) in any accuracy but also is mathematically meaningless because a solution of (14) simply does not

exist in this case. This is why we state that any solutions based on the TWE (4) using the wind profile (2) that violates the solvability condition are not mathematically or physically meaningful.

## 6 SUMMARY AND REMARKS

It should be emphasized that accurate interpretation of the even zonal gravitational coefficients ( $J_{2k} + \Delta J_{2k}$ ) with  $k \geq 1$  in (1) expected from the Juno mission will be a difficult task. This is because both the effects of rotation and the equatorially symmetric zonal winds make contributions: isolating a small wind-induced correction  $\Delta J_{2k}$  from the observed ( $J_{2k} + \Delta J_{2k}$ ) is challenging. By contrast, the odd gravitational coefficients ( $J_{2k+1} + \Delta J_{2k+1}$ ) with  $k \geq 1$  in (1), since  $J_{2k+1} = 0$  due to the equatorial symmetry, directly reflect the structure and amplitude of the equatorially anti-symmetric winds. It is therefore crucial that one establishes a valid relationship between the physically possible profile of Jovian equatorially antisymmetric winds and the odd zonal gravitational coefficients  $\Delta J_{2k+1}$  that can then be used to interpret the anticipated gravitational measurements from the Juno mission.

In this paper, we have shown that the method, adopted by Kaspi (2013), based on the TWE (4) using the zonal winds (2), violates the required solvability condition for the TGWE (3) and, hence, cannot provide mathematically and physically meaningful results. Based on the TGWE (3) and the zonal winds (13), we propose a new method based on an antisymmetric wind profile that satisfies the required solvability condition, obeys the equatorial symmetry condition and is consistent with the cloud-level zonal winds of Jupiter. Though we have only taken the leading-order term in the general expansion (12), it is possible to solve an inverse problem by including higher-order terms in (12) when the measurements of the Jovian antisymmetric gravitational field from the Juno mission become available in the future.

The present study also reinforces the view that the TWE (4) is fundamentally different from the TGWE (3). The extra integral term in (3) is not only generally of the same order of magnitude as the other terms and, hence, must be retained, but it also alters the mathematical nature of the governing equation in a profound way. With the extra kernel integral term, the solvability condition must be satisfied in order that solutions to the TGWE (3) can exist while the TWE (4) – which is supposed to provide an approximation to the TGWE (3) – does not have such a requirement. Our method of computing the Jovian odd gravitational coefficients using the idealized zonal winds (13) in this paper offers a promising way of interpreting the equatorially antisymmetric gravitational data expected from the Juno mission.

While this paper was in review, a paper appeared by Galanti et al. (2017). They extended the analysis of Zhang et al. (2015) in spherical geometry to that of spheroidal geometry, suggesting that the dominant balance is described by that of the simplified thermal wind approach. Their analysis is, however, only concerned with the equatorially symmetric flow while this study is concerned with the equatorially anti-symmetric flow.

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